

RIEFFEL DEFORMATION OF TENSOR FUNCTOR AND BRAIDED QUANTUM GROUPS

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ABSTRACT. In this paper we apply Rieffel deformation to the C^* -tensor product viewed as a functor in the category of C^* -algebras with an abelian group action. In the case of the Rieffel deformation of a quantum group with the action by automorphisms the deformed tensor product enables us to view the deformed object as a braided quantum group. We use this construction as a hint towards a braided quantum groups theory. In particular we construct a bicharacter together with the dual braided quantum group. We employ our method to get a braided quantum Minkowski space. Its description in terms of the deformed space-time coordinates is provided.

1. INTRODUCTION

The theory of locally compact quantum groups (LCQG) is well established by now. For the axiomatic formulations of LCQG with the existence of Haar measure postulated we refer the reader to [5] or [7]. For the theory with the multiplicative unitary playing central role we refer to [12]. Roughly speaking a quantum group is a pair $\mathbb{G} = (A, \Delta)$ where A is a C^* -algebra and $\Delta \in \text{Mor}(A, A \otimes A)$. Let us postpone the discussion of morphism and multipliers to Section 2 where the respective \mathcal{C}^* -category will be introduced and focus on the tensor product $A \otimes B$ of C^* -algebras A and B .

Remarkably $A \otimes B$ is not a uniquely defined C^* -algebra. Between the minimal (spatial) and the maximal (universal) tensor products $A \otimes_{\min} B$, $A \otimes_{\max} B$ we have a whole family of alternative tensor products $A \otimes_{\lambda} B$ unless $A \otimes_{\min} B = A \otimes_{\max} B$. In this paper we restrict our interest to the spatial tensor product thus we ignore min in the notation writing $A \otimes B$. For an enjoyable book with the tensor product of C^* -algebras intelligibly explained we refer to [1].

Let A and B be faithfully represented on Hilbert spaces H and K respectively. Then $A \otimes B$ is defined as the closed linear span of $\{a \otimes b \in B(H \otimes K) : a \in A, b \in B\}$. Remarkably the resulting C^* -algebra $A \otimes B$ does not depend on the choice of faithful representations. The tensor product construction lifts to the morphisms level: for $\pi \in \text{Mor}(A_1, B_1)$ and $\sigma \in \text{Mor}(A_2, B_2)$ there exists a unique morphism $\pi \otimes \sigma \in \text{Mor}(A_1 \otimes B_1, A_2 \otimes B_2)$ satisfying $(\pi \otimes \sigma)(a \otimes b) = \pi(a) \otimes \sigma(b)$ for any $a \in A, b \in B$. Summarizing we get the associative functor $\otimes : \mathcal{C}^* \times \mathcal{C}^* \rightarrow \mathcal{C}^*$

$$\otimes \circ (\otimes \times \text{id}) = \otimes \circ (\text{id} \times \otimes)$$

together with a pair of distinguished morphisms $\iota^A \in \text{Mor}(A, A \otimes B)$ and $\iota^B \in \text{Mor}(B, A \otimes B)$

$$\iota^A : A \ni a \mapsto a \otimes \mathbf{1} \in M(A \otimes B)$$

$$\iota^B : B \ni b \mapsto \mathbf{1} \otimes b \in M(A \otimes B)$$

satisfying $A \otimes B = [\iota^A(A) \cdot \iota^B(B)]$ for $[X]$ - notation we refer to the last paragraph of this section.

The tensor product functor has the covariant version defined in the category \mathcal{C}_G^* of C^* -algebras acted by a locally compact group G which we discuss in Section 2. An object of \mathcal{C}_G^* is a pair (A, ρ^A) where $\rho^A : G \rightarrow \text{Aut}(A)$ is an action of G on A and the action of G on $A \otimes B$ is given by $\rho_g^{A \otimes B} = \rho_g^A \otimes \rho_g^B$.

Let Γ be an abelian group and Ψ a 2-cocycle on the Pontryagin dual $\hat{\Gamma}$. In Section 3 we view Rieffel deformation \mathcal{RD}^Ψ as a functor $\mathcal{RD}^\Psi : \mathcal{C}_\Gamma^* \rightarrow \mathcal{C}_\Gamma^*$. Observing the invertibility of \mathcal{RD}^Ψ

$$(\mathcal{RD}^\Psi)^{-1} = \mathcal{RD}^{\bar{\Psi}}$$

we define a deformed tensor functor $\otimes_\Psi : \mathcal{C}_\Gamma^* \times \mathcal{C}_\Gamma^* \rightarrow \mathcal{C}_\Gamma^*$ by the formula

$$\otimes_\Psi = \mathcal{RD}^\Psi \circ \otimes \circ (\mathcal{RD}^{\overline{\Psi}} \times \mathcal{RD}^{\overline{\Psi}})$$

We show that \otimes_Ψ hereditates all properties of \otimes except the commutativity of $A \otimes_\Psi 1$ and $1 \otimes_\Psi B$ inside $M(A \otimes_\Psi B)$.

Having the functor \otimes_Ψ defined we move on in Section 4 to the \otimes_Ψ - braided quantum groups. We observe that a quantum group $\mathbb{G} = (A, \Delta)$ on which Γ acts by quantum group automorphisms gives rise by Rieffel deformation to a braided quantum group $\mathbb{G}^\Psi = (A^\Psi, \Delta^\Psi)$ with $\Delta^\Psi \in \text{Mor}(A^\Psi, A^\Psi \otimes_\Psi A^\Psi)$ satisfying the coassociativity and cancellation law. Remarkably we get a candidates for bicharacter W^Ψ of \mathbb{G}^Ψ and the dual braided quantum group where the braiding for the dual object is implemented by the flip of Ψ .

In the last section we apply our theory to construct a braided quantum Minkowski space. Its description in terms of the deformed space-time coordinates $(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$ is provided. The comultiplication Δ^Ψ when applied to the coordinates is shown to give $\Delta^\Psi(\hat{x}_i) = \hat{x}_i \otimes_\Psi \mathbb{1} + \mathbb{1} \otimes_\Psi \hat{x}_i$, for $i = 0, 1, 2, 3$.

Some remarks about the notation. For a subset X of a Banach space B , X^{cls} denotes the closed linear span of X . Alternatively we shall write $[X] = X^{\text{cls}}$. The (Banach) dual A' of a C^* -algebra A is an A -bimodule where for $\omega \in A'$ and $b, c \in A$ we define $b \cdot \omega \cdot c$ by the formula:

$$(b \cdot \omega \cdot c)(a) = \omega(cab)$$

for any $a \in A$. The C^* -algebra of multipliers of A is denoted by $M(A)$. $M(A)$ is equipped with the strict topology - for the discussion of natural C^* -topologies we refer to [15].

2. \mathcal{C}_Γ^* -CATEGORY AND THE CROSSED PRODUCT FUNCTOR

Since the results of this paper are naturally expressed in the category theory terms let us first introduce the category \mathcal{C}^* of C^* -algebras with Woronowicz morphisms. For $A, B \in \text{Obj}(\mathcal{C}^*)$, a morphism $\pi : A \rightarrow B$ is a $*$ -homomorphism $\pi : A \rightarrow M(B)$ which is non-degenerate $\overline{\pi(A)B}^{\|\cdot\|} = B$. One may introduce the unique extension $\bar{\pi} : M(A) \rightarrow M(B)$ of π satisfying $\bar{\pi}(a_1)\pi(a_2)b = \pi(a_1a_2)b$. Denoting the extension by the same symbol π we have a well defined morphisms composition: $\sigma \circ \pi \in \text{Mor}(A, C)$ for any $\pi \in \text{Mor}(A, B)$ and $\sigma \in \text{Mor}(B, C)$.

Let Γ be an abelian locally compact group. We use the additive notation $\gamma + \gamma' \in \Gamma$ for any $\gamma, \gamma' \in \Gamma$. The category of Γ - C^* -algebras is denoted by \mathcal{C}_Γ^* . We write $(A, \rho^A) \in \text{Obj}(\mathcal{C}_\Gamma^*)$ where A is a C^* algebra and $\rho^A : \Gamma \rightarrow \text{Aut}(A)$ is a continuous action, i.e. $\rho_{\gamma+\gamma'}^A = \rho_\gamma^A \circ \rho_{\gamma'}^A$ and the map

$$\Gamma \ni \gamma \mapsto \rho_\gamma(a) \in A$$

is norm continuous for any $a \in A$. A morphism $\pi : (A, \rho^A) \rightarrow (B, \rho^B)$ is a morphism $\pi \in \text{Mor}(A, B)$ which is covariant

$$\rho_\gamma^B \circ \pi = \pi \circ \rho_\gamma^A.$$

We write Mor_Γ and $\pi \in \text{Mor}_\Gamma(A, B)$ avoiding to put ρ^A, ρ^B under the Mor_Γ -symbol.

An element $(A, \rho^A) \in \text{Obj}(\mathcal{C}_\Gamma^*)$ gives rise to the crossed product C^* -algebra $\Gamma \ltimes A \in \text{Obj}(\mathcal{C}^*)$. For the crossed product theory we refer to [13] recalling only the essential properties of $\Gamma \ltimes A$. Remarkably A and $C^*(\Gamma)$ embed into $M(\Gamma \ltimes A)$ with

$$\begin{aligned} \iota^A &\in \text{Mor}(A, \Gamma \ltimes A) \\ \iota^{C^*(\Gamma)} &\in \text{Mor}(C^*(\Gamma), \Gamma \ltimes A) \end{aligned}$$

and identifying $A, C^*(\Gamma)$ with the subalgebras of $M(\Gamma \ltimes A)$ we have

$$\Gamma \ltimes A = [C^*(\Gamma) \cdot A]$$

Definition 2.1. Let $(A, \rho^A) \in \text{Obj}(\mathcal{C}_\Gamma^*)$ and let H be a Hilbert space. Let $\pi : A \rightarrow B(H)$ be a non-degenerate representation of A on H and $\theta : \Gamma \rightarrow B(H)$ a unitary strongly continuous representation of Γ on H . We say that (θ, π) is a covariant representation of (A, ρ^A) if for any $a \in A$ and $\gamma \in \Gamma$ we have

$$\pi(\rho_\gamma^A(a)) = \theta_\gamma \pi(a) \theta_\gamma^*.$$

Let (θ, π) be a covariant representation of (A, ρ^A) on a Hilbert space H . The unique extension of $\theta : \Gamma \rightarrow B(H)$ to the C^* -representation satisfying

$$M(C^*(\Gamma)) \ni \lambda_\gamma \mapsto \theta_\gamma \in B(H)$$

will be denoted by the same symbol $\theta : C^*(\Gamma) \rightarrow B(H)$. The universal property of $\Gamma \ltimes A$ is formulated as a 1-1 correspondence between covariant representations of (A, ρ^A) and non-degenerate representation of $\Gamma \ltimes A$ where the representation assigned to (θ, π) denoted by $\theta \ltimes \pi$ is characterized as the unique representation satisfying

$$\begin{aligned} (\theta \ltimes \pi) \circ \iota^A &= \pi \circ \iota^A \\ (\theta \ltimes \pi) \circ \iota^{C^*(\Gamma)} &= \theta \end{aligned}$$

The universal property of $\Gamma \ltimes A$ for covariant representations of $(A, \rho^A) \in \text{Obj}(\mathcal{C}_\Gamma^*)$ has the covariant morphisms version.

Definition 2.2. Let $(A, \rho^A) \in \text{Obj}(\mathcal{C}_\Gamma^*)$, $C \in \text{Obj}(\mathcal{C}^*)$, $\pi \in \text{Mor}(A, C)$ and $\theta : \Gamma \rightarrow M(C)$ a unitary strictly continuous representation. We say that (θ, π) is a covariant morphism of (A, ρ^A) if for any $a \in A$ and $\gamma \in \Gamma$ we have

$$\pi(\rho_\gamma^A(a)) = \theta_\gamma \pi(a) \theta_\gamma^*.$$

For a covariant morphism (θ, π) there exists $\theta \ltimes \pi \in \text{Mor}(\Gamma \ltimes A, C)$ uniquely characterized by

$$\begin{aligned} (\theta \ltimes \pi) \circ \iota^A &= \pi \circ \iota^A \\ (\theta \ltimes \pi) \circ \iota^{C^*(\Gamma)} &= \theta \end{aligned}$$

Let $\widehat{\Gamma}$ be the Pontryagin dual of Γ with the duality

$$\widehat{\Gamma} \times \Gamma \ni (\widehat{\gamma}, \gamma) \mapsto \langle \widehat{\gamma}, \gamma \rangle \in \mathbb{T}^1$$

and let $\widehat{\gamma} \in \widehat{\Gamma}$. Noting that $\Gamma \ni \gamma \mapsto \langle \widehat{\gamma}, \gamma \rangle \lambda_\gamma \in M(\Gamma \ltimes A)$ is a representation such that the pair $(\langle \widehat{\gamma}, \cdot \rangle \lambda, \iota^A)$ is a covariant morphism and using universality of $\Gamma \ltimes A$ we get a morphism $\widehat{\rho}_{\widehat{\gamma}} \in \text{Mor}(\Gamma \ltimes A, \Gamma \ltimes A)$. Since $\rho_{\widehat{\gamma} + \widehat{\gamma}'} = \widehat{\rho}_{\widehat{\gamma}} \circ \widehat{\rho}_{\widehat{\gamma}'}$ we may see that $\widehat{\rho}_{\widehat{\gamma}} \in \text{Aut}(\Gamma \ltimes A)$ and we get the famous dual action $\widehat{\rho}$ of $\widehat{\Gamma}$ on $\Gamma \ltimes A$.

Let $\pi \in \text{Mor}_\Gamma(A, B)$. Again, applying the universal property of $\Gamma \ltimes A$ in the context of the covariant morphism (λ, π) we get $\lambda \ltimes \pi \in \text{Mor}(\Gamma \ltimes A, \Gamma \ltimes B)$. Its covariance with respect to the dual actions is clear and enables us to view the crossed product as a functor $\mathcal{CP} : \mathcal{C}_\Gamma^* \rightarrow \mathcal{C}_{\widehat{\Gamma}}^*$

$$\begin{aligned} \mathcal{CP}((A, \rho^A)) &= (\Gamma \ltimes A, \widehat{\rho}) \\ \mathcal{CP}(\pi) &= \lambda \ltimes \pi \end{aligned}$$

Remarkably we have a characterization of $(D, \widehat{\rho}^D) \in \text{Obj}(\mathcal{C}_{\widehat{\Gamma}}^*)$ that are of the crossed product form. Note that for $(\Gamma \ltimes A, \widehat{\rho}) \in \text{Obj}(\mathcal{C}_{\widehat{\Gamma}}^*)$ we have a representation $\lambda : \Gamma \rightarrow M(\Gamma \ltimes A)$ and

$$\widehat{\rho}_{\widehat{\gamma}}(\lambda_\gamma) = \langle \widehat{\gamma}, \gamma \rangle \lambda_\gamma.$$

As was shown by Landstad [6], a $\widehat{\Gamma}$ - C^* -algebra $(D, \widehat{\rho}^D) \in \text{Obj}(\mathcal{C}_{\widehat{\Gamma}}^*)$ that is equipped with a representation $\lambda : \Gamma \rightarrow M(D)$ satisfying

$$\widehat{\rho}_{\widehat{\gamma}}^D(\lambda_\gamma) = \langle \widehat{\gamma}, \gamma \rangle \lambda_\gamma$$

may be identified with a crossed product of a certain (essentially unique) (A, ρ^A) . Extending λ to an injective morphism $\lambda \in \text{Mor}(C^*(\Gamma), D)$, we characterize $A \subset M(D)$ as the Γ - C^* -algebra satisfying Landstad conditions

$$A = \left\{ d \in M(D) \left| \begin{array}{l} 1. \widehat{\rho}_{\widehat{\gamma}}^D(d) = d \\ 2. \text{The map } \Gamma \ni \gamma \mapsto \lambda_\gamma d \lambda_\gamma^* \in M(D) \\ \quad \text{is norm-continuous} \\ 3. xdy \in D \text{ for any } x, y \in C^*(\Gamma) \end{array} \right. \right\} \quad (2.1)$$

where ρ^A is implemented by λ

$$\rho_\gamma^A(a) = \lambda_\gamma a \lambda_\gamma^*.$$

A triple $(D, \hat{\rho}^D, \lambda)$ will be called a Γ -product and $A \subset M(D)$ a Landsat algebra of the corresponding Γ -product.

Remark 2.3. Let H be a Hilbert space and $A \subset B(H)$. Then there exists a faithful representation $\Gamma \ltimes A$ on $L^2(\Gamma) \otimes H$ that corresponds to a unique covariant representation (θ, π) . Identifying $L^2(\Gamma) \otimes H$ with the Hilbert space of square integrable maps from Γ to H $L^2(\Gamma) \otimes H \cong L^2(\Gamma, H)$ we have

$$\begin{aligned} (\theta_\gamma x)(\gamma') &= x(\gamma' + \gamma) \\ (\pi(a)x)(\gamma) &= \rho_\gamma^A(a)x(\gamma) \end{aligned}$$

for any $x \in L^2(\Gamma, H)$ and $\gamma, \gamma' \in \Gamma$.

3. RIEFFEL DEFORMATION FUNCTOR

Rieffel deformation as formulated in this section was introduced in [3] - here we rephrase it in categorical terms. For the original approach developed by M. Rieffel we refer to [10].

Let $\Psi : \hat{\Gamma} \times \hat{\Gamma} \rightarrow \mathbb{T}^1$ be a continuous 2-cocycle on Ψ

- (i) $\Psi(e, \hat{\gamma}) = \Psi(\hat{\gamma}, e) = 1$ for all $\hat{\gamma} \in \hat{\Gamma}$
- (ii) $\Psi(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_2, \hat{\gamma}_3) = \Psi(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3)\Psi(\hat{\gamma}_1, \hat{\gamma}_2)$ for all $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3 \in \hat{\Gamma}$

Note that the set $\mathcal{COC}_2(\hat{\Gamma})$ of 2-cocycles on $\hat{\Gamma}$ forms a multiplicative group and $\Psi^{-1} = \overline{\Psi}$. For any $\hat{\gamma} \in \hat{\Gamma}$ we define an auxiliary function $\Psi_{\hat{\gamma}} : \hat{\Gamma} \rightarrow \mathbb{T}$ where

$$\Psi_{\hat{\gamma}}(\hat{\gamma}') = \Psi(\hat{\gamma}', \hat{\gamma})$$

- its role in the theory will be explained later.

In this section we shall write (A, ρ) ignoring A in ρ^A . For $(A, \rho) \in \text{Obj}(\mathcal{C}_\Gamma^*)$ we form the crossed product $\Gamma \ltimes A$. Identifying $C^*(\Gamma)$ with $C_0(\hat{\Gamma})$ we get a unitary family $U_{\hat{\gamma}} = \iota^{C^*(\Gamma)}(\Psi_{\hat{\gamma}}) \in M(\Gamma \ltimes A)$. For any $\hat{\gamma} \in \Gamma$ we define $\hat{\rho}_{\hat{\gamma}}^\Psi \in \text{Aut}(\Gamma \ltimes A)$ by the formula

$$\hat{\rho}_{\hat{\gamma}}^\Psi(b) = U_{\hat{\gamma}}^* \hat{\rho}_{\hat{\gamma}}(b) U_{\hat{\gamma}} \quad (3.1)$$

As was shown in Theorem 3.1 of [3], $\hat{\rho}_{\hat{\gamma}}^\Psi$ defines a continuous action of $\hat{\Gamma}$ on $\Gamma \ltimes A$ and the triple $(\Gamma \ltimes A, \hat{\rho}_{\hat{\gamma}}^\Psi, \lambda)$ is a Γ -product. The Landstad algebra of this Γ -triple is called a Rieffel deformation of A and denoted A^Ψ . Remarkably A^Ψ being the Landstad algebra of $(\Gamma \ltimes A, \hat{\rho}_{\hat{\gamma}}^\Psi, \lambda)$ is equipped with Γ -action that is implemented by λ :

$$A \ni a \mapsto \lambda_\gamma a \lambda_\gamma^* \in A$$

for any $a \in A^\Psi$ which we denote by $\rho : \Gamma \rightarrow \text{Aut}(A^\Psi)$ and we have $\Gamma \ltimes A^\Psi = \Gamma \ltimes A$.

Let $\pi \in \text{Mor}_\Gamma(A, B)$ and $\lambda \ltimes \pi \in \text{Mor}_{\hat{\Gamma}}(\Gamma \ltimes A, \Gamma \ltimes B)$. It was shown in [3] that $(\lambda \ltimes \pi)(A^\Psi) \subset M(B^\Psi)$ and π gives rise to a morphism $\pi^\Psi \in \text{Mor}_\Gamma(A^\Psi, B^\Psi)$ where $\pi^\Psi = \lambda \ltimes \pi|_{A^\Psi}$. Since for $\pi \in \text{Mor}_\Gamma(A, B)$ and $\sigma \in \text{Mor}_\Gamma(B, C)$ we have

$$\lambda \ltimes (\sigma \circ \pi) = (\lambda \ltimes \sigma) \circ (\lambda \ltimes \pi)$$

we get $(\sigma \circ \pi)^\Psi = \sigma^\Psi \circ \pi^\Psi$.

Definition 3.1. Let Γ be an abelian group. For any $\Psi \in \mathcal{COC}_2(\hat{\Gamma})$ we define a functor

$$\mathcal{RD}^\Psi : \mathcal{C}_\Gamma^* \rightarrow \mathcal{C}_\Gamma^*$$

such that adopting the above notation we have

$$\begin{aligned} \mathcal{RD}^\Psi(A, \rho) &= (A^\Psi, \rho) \\ \mathcal{RD}^\Psi(\pi) &= \pi^\Psi \end{aligned}$$

which we call a Rieffel deformation functor associated to Ψ .

For any category \mathcal{C} , a functor $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{C}$ is called an endofunctor. An endofunctor \mathcal{R} having an inverse is an autofunctor. The class of autofunctors of a category \mathcal{C} is denoted by $\mathcal{AFUN}(\mathcal{C})$.

Remark 3.2. Noting that for $\Psi, \Phi \in \mathcal{COC}_2(\widehat{\Gamma})$ we have $A^{\Psi \cdot \Phi} = (A^\Psi)^\Phi$ and $\pi^{\Psi \cdot \Phi} = (\pi^\Psi)^\Phi$ it is tempting to say that the map $\mathcal{RD} : \mathcal{COC}_2(\widehat{\Gamma}) \rightarrow \mathcal{RD}^\Psi \in \mathcal{AFUN}(\mathcal{C}_\Gamma^*)$ is a group homomorphism:

$$\mathcal{RD}^{\Psi\Phi} = \mathcal{RD}^\Psi \circ \mathcal{RD}^\Phi.$$

Since we do not know whether $\mathcal{AFUN}(\mathcal{C}_\Gamma^*)$ forms a set (in the sense of the set theory) we carefully claim that the set of autofunctors $\{\mathcal{RD}^\Psi : \Psi \in \mathcal{COC}_2(\widehat{\Gamma})\}$ is closed under the composition and $\mathcal{RD}^{\Psi\Phi} = \mathcal{RD}^\Psi \circ \mathcal{RD}^\Phi$. Since $\Psi^{-1} = \bar{\Psi}$ we have

$$(\mathcal{RD}^\Psi)^{-1} = \mathcal{RD}^{\bar{\Psi}} \quad (3.2)$$

Let us mention that \mathcal{RD}^Ψ is exact (see Proposition 2.9 of [3]). In particular $\mathcal{RD}^\Psi(\pi)$ is injective if (and only if by invertibility of \mathcal{RD}^Ψ) π is injective.

3.1. Rieffel deformation of tensor product. Let $(A, \rho^A), (B, \rho^B) \in \text{Obj}(\mathcal{C}_\Gamma^*)$. Defining the action $\rho^{A \otimes B}$ on $A \otimes B$

$$\rho_\gamma^{A \otimes B}(a \otimes b) = \rho_\gamma^A(a) \otimes \rho_\gamma^B(b)$$

we get the tensor functor $\otimes : \mathcal{C}_\Gamma^* \times \mathcal{C}_\Gamma^* \rightarrow \mathcal{C}_\Gamma^*$. Clearly the functor \otimes is associative:

$$\otimes \circ (\otimes \times \text{id}) = \otimes \circ (\text{id} \times \otimes)$$

Moreover for $(A, \rho^A), (B, \rho^B) \in \text{Obj}(\mathcal{C}_\Gamma^*)$ we have a pair of injective morphisms

$$\iota^A \in \text{Mor}_\Gamma(A, A \otimes B) : \iota^A(a) = a \otimes 1$$

$$\iota^B \in \text{Mor}_\Gamma(B, A \otimes B) : \iota^B(b) = 1 \otimes b$$

and we have $A \otimes B = [\iota^A(A) \cdot \iota^B(B)]$.

Definition 3.3. Let $\Psi \in \mathcal{COC}_2(\widehat{\Gamma})$. We define $\otimes_\Psi : \mathcal{C}_\Gamma^* \times \mathcal{C}_\Gamma^* \rightarrow \mathcal{C}_\Gamma^*$

$$\otimes_\Psi = \mathcal{RD}^\Psi \circ \otimes \circ (\mathcal{RD}^{\bar{\Psi}} \times \mathcal{RD}^{\bar{\Psi}})$$

Note that

$$A^\Psi \otimes_\Psi B^\Psi = (A \otimes B)^\Psi \quad (3.3)$$

Since $\iota^A \in \text{Mor}_\Gamma(A, A \otimes B)$ we immediately see that $\mathcal{RD}^\Psi(\iota^A) \in \text{Mor}_\Gamma(A^\Psi, A^\Psi \otimes_\Psi B^\Psi)$ is an injective morphism. Similarly, for ι^B we get injective $\mathcal{RD}^\Psi(\iota^B) \in \text{Mor}_\Gamma(B^\Psi, A^\Psi \otimes_\Psi B^\Psi)$. We shall use a notation

$$\begin{aligned} \mathcal{RD}^\Psi(\iota^A)(a) &= a \otimes_\Psi 1 \\ \mathcal{RD}^\Psi(\iota^B)(b) &= 1 \otimes_\Psi b \end{aligned} \quad (3.4)$$

for $a \in A^\Psi$ and $b \in B^\Psi$.

Proposition 3.4. *Adopting the above notation the functor \otimes_Ψ is associative $\otimes_\Psi \circ (\otimes_\Psi \times \text{id}) = \otimes_\Psi \circ (\text{id} \times \otimes_\Psi)$. Moreover $A^\Psi \otimes_\Psi B^\Psi = [(A^\Psi \otimes_\Psi 1) \cdot (1 \otimes_\Psi B^\Psi)]$.*

Proof. Let $A, B, C \in \mathcal{C}_\Gamma^*$. We have

$$\begin{aligned} (A^\Psi \otimes_\Psi B^\Psi) \otimes_\Psi C^\Psi &= (A \otimes B)^\Psi \otimes_\Psi C^\Psi \\ &= ((A \otimes B) \otimes C)^\Psi \\ &= (A \otimes (B \otimes C))^\Psi = A^\Psi \otimes_\Psi (B^\Psi \otimes_\Psi C^\Psi) \end{aligned}$$

Since \mathcal{RD}^Ψ is an invertible functor and all C^* algebras are of the form A^Ψ we get associativity of \otimes_Ψ . In order to prove the second part of the proposition we use Lemma 3.4 of [4] in the context of $\iota^A(A), \iota^B(B) \subset M(A \otimes B)$ getting $A^\Psi \otimes_\Psi B^\Psi = [(A^\Psi \otimes_\Psi 1) \cdot (1 \otimes_\Psi B)]$. \square

Remark 3.5. Applying \otimes_Ψ -functor to $\pi_1 \in \text{Mor}_\Gamma(A_1, B_1)$ and $\pi_2 \in \text{Mor}_\Gamma(A_2, B_2)$ we get $\pi_1 \otimes_\Psi \pi_2 \in \text{Mor}_\Gamma(A_1 \otimes_\Psi A_2, B_1 \otimes_\Psi B_2)$. In particular for $C \in \text{Obj}(\mathcal{C}_\Gamma^*)$ and $\pi \in \text{Mor}_\Gamma(A, B)$ we have $\pi \otimes_\Psi \text{id} \in \text{Mor}_\Gamma(A \otimes_\Psi C, B \otimes_\Psi C)$.

4. RIEFFEL DEFORMATION AND BRAIDED QUANTUM GROUPS

In this section we apply Rieffel deformation \mathcal{RD}^Ψ to a quantum group \mathbb{G} with Γ acting by automorphisms of \mathbb{G} . We adopt Definition 2.3 [12] and use multiplicative unitary W as the main object of the quantum groups theory. In particular Haar weights are not discussed in this section. For the theory of manageable multiplicative unitaries we refer to [16] and for the theory of modular multiplicative unitaries we refer to [11].

Definition 4.1 (Definition 2.3 [12]). Let A be a C^* -algebra and $\Delta \in \text{Mor}(A, A \otimes A)$. We say that a pair $\mathbb{G} = (A, \Delta)$ is a quantum group if there exists a Hilbert space H and a modular multiplicative unitary $W \in B(H \otimes H)$ such that (A, Δ) is isomorphic to the C^* -algebra with comultiplication associated to W . In such a case we shall say that W is a modular multiplicative unitary giving rise to the quantum group \mathbb{G} .

Let \mathbb{G} be a quantum group with a modular multiplicative unitary $W \in B(H \otimes H)$. When convenient we write $A = C_0(\mathbb{G})$ and $H = L^2(\mathbb{G})$ ignoring the fact that H does not have to correspond to the GNS Hilbert space for the Haar weight of \mathbb{G} . A quantum group \mathbb{G} has a dual quantum group $\widehat{\mathbb{G}}$. The modular multiplicative unitary of $\widehat{\mathbb{G}}$ is given by $\widehat{W} = \Sigma W^* \Sigma$ where $\Sigma : H \otimes H \rightarrow H \otimes H$ is the flip operator: $\Sigma(x \otimes y) = y \otimes x$ for $x, y \in H$. Note that $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$.

Let \mathbb{H} and \mathbb{G} be locally compact quantum groups. In order to introduce a concept of a quantum group homomorphism $\pi : \mathbb{H} \rightarrow \mathbb{G}$ we need the universal objects $\mathbb{H}^u = (C_0^u(\mathbb{H}), \Delta_{\mathbb{H}}^u)$ and $\mathbb{G}^u = (C_0^u(\mathbb{G}), \Delta_{\mathbb{G}}^u)$ assigned to \mathbb{H} and \mathbb{G} respectively. For the definition of the universal version of a quantum group given by a multiplicative unitary W we refer to Definition 5.1 of [12]. Let us emphasize that the universal version of a quantum group is not a quantum group in the sense of Definition 4.1. The concept of quantum groups homomorphism in the framework of Definition 4.1 was developed in [8].

Definition 4.2. Let \mathbb{H} and \mathbb{G} be quantum groups and $\pi \in \text{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$. We say that π is a strong quantum homomorphism and write $\pi : \mathbb{H} \rightarrow \mathbb{G}$ if

$$\Delta_{\mathbb{H}}^u \circ \pi = (\pi \otimes \pi) \circ \Delta_{\mathbb{G}}^u$$

When convenient we refer to a strong quantum homomorphism $\pi : \mathbb{H} \rightarrow \mathbb{G}$ as homomorphism ignoring the strong and quantum adjectives.

Remark 4.3. Let $\mathbb{H} = (C_0(\mathbb{H}), \Delta_{\mathbb{H}})$ and $\mathbb{G} = (C_0(\mathbb{G}), \Delta_{\mathbb{G}})$ be quantum groups and $\pi : \mathbb{H} \rightarrow \mathbb{G}$. There is a well defined homomorphisms duality such that $\widehat{\widehat{\pi}} : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{H}}$ and $\widehat{\widehat{\pi}} = \pi$. For a quantum group \mathbb{K} and a homomorphism $\sigma : \mathbb{G} \rightarrow \mathbb{K}$ we have $\widehat{\sigma \circ \pi} = \widehat{\pi} \circ \widehat{\sigma}$. In particular $\pi : \mathbb{H} \rightarrow \mathbb{G}$ is invertible if and only if $\widehat{\pi}$ is invertible. An isomorphism is homomorphism which is invertible and an isomorphism $\pi : \mathbb{G} \rightarrow \mathbb{G}$ is called an automorphism. The set of automorphism is denoted by $\text{Aut}(\mathbb{G})$.

Let $\pi : \mathbb{H} \rightarrow \mathbb{G}$ be an isomorphism. As was shown in [2] Theorem 1.10, π descends to the C^* -isomorphism $\pi_r : C_0(\mathbb{G}) \rightarrow C_0(\mathbb{H})$ such that

$$\Delta_{\mathbb{H}} \circ \pi_r = (\pi_r \otimes \pi_r) \circ \Delta_{\mathbb{G}}.$$

In particular discussing the automorphism group $\text{Aut}(\mathbb{G})$, the universal version \mathbb{G}^u is irrelevant. Let $\pi \in \text{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}))$ be an automorphism of \mathbb{G} and W the modular multiplicative unitary viewed as $W \in M(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$. The dual automorphism $\widehat{\pi} : C_0(\widehat{\mathbb{G}}) \rightarrow C_0(\widehat{\mathbb{G}})$ is uniquely characterized by the equality

$$(\widehat{\pi} \otimes \text{id})W = (\text{id} \otimes \pi)W \quad (4.1)$$

Remark 4.4. Let $\mathbb{G} = (A, \Delta)$ be a quantum group. Then $\text{Aut}(\mathbb{G})$ equipped with composition forms a group. Equipping $\text{Aut}(\mathbb{G})$ with the weakest topology such that for all $a \in A$ the mappings

$$\text{Aut}(\mathbb{G}) \ni \pi \rightarrow \pi(a) \in A$$

are norm-continuous, we view $\text{Aut}(\mathbb{G})$ as a topological group. Noting that the automorphisms duality $\text{Aut}(\mathbb{G}) \ni \pi \mapsto \widehat{\pi} \in \text{Aut}(\widehat{\mathbb{G}})$ is composition reversing $\widehat{\sigma \circ \pi} = \widehat{\pi} \circ \widehat{\sigma}$ we see that the map

$$\text{Aut}(\mathbb{G}) \ni \pi \mapsto \widehat{\pi} \in \text{Aut}(\widehat{\mathbb{G}})^{\text{op}} \quad (4.2)$$

is a groups isomorphism. The relation $(\widehat{\pi} \otimes \text{id})W = (\text{id} \otimes \pi)W$ enables us to prove that the map (4.2) is a topological groups isomorphism.

Definition 4.5. Let $\mathbb{G} = (A, \Delta)$ be a quantum group, Γ an abelian group and $(A, \rho^A) \in \text{Obj}(\mathcal{C}_\Gamma^*)$. If $\rho_\gamma^A \in \text{Aut}(\mathbb{G})$ then we say that Γ acts on A by automorphisms.

Let ρ^A be an action of Γ on A and $\mathbb{G} = (A, \Delta)$. Then ρ^A is an action by automorphism if and only if $\Delta \in \text{Mor}_\Gamma(A, A \otimes A)$. Clearly $\Gamma \ni \gamma \mapsto \rho_\gamma \in \text{Aut}(\mathbb{G})$ is a continuous homomorphism. For $\gamma \in \Gamma$ we may consider the dual morphism $\widehat{\rho}_\gamma^A \in \text{Aut}(\widehat{\mathbb{G}})$ (should not be confused with the dual action $\widehat{\rho}_{\widehat{\gamma}}$). Since Γ is abelian the map $\Gamma \ni \gamma \mapsto \widehat{\rho}_\gamma^A \in \text{Aut}(\text{C}_0(\widehat{\mathbb{G}}))$ is an action of Γ on $\widehat{\mathbb{G}}$ by automorphisms. In what follows $\widehat{\rho}^A$ will be denoted by $\rho^{\widehat{A}}$.

4.1. Rieffel deformation of a quantum group with an action of Γ by automorphisms.

Let $\mathbb{G} = (A, \Delta)$ be a quantum group with an action ρ of Γ by automorphisms and let Ψ be a 2-cocycle on $\widehat{\Gamma}$. Applying \mathcal{RD}^Ψ to A we get a C^* -algebra A^Ψ . In this section we show that A^Ψ is equipped with a braided comultiplication $\Delta^\Psi \in \text{Mor}_\Gamma(A^\Psi, A^\Psi \otimes_\Psi A^\Psi)$ which is coassociative and satisfies cancellation law. Moreover assuming Ψ to be a bicharacter (which we do in what follows)

$$\begin{aligned}\Psi(\widehat{\gamma}_1, \widehat{\gamma}_2 + \widehat{\gamma}_3) &= \Psi(\widehat{\gamma}_1, \widehat{\gamma}_2)\Psi(\widehat{\gamma}_1, \widehat{\gamma}_3) \\ \Psi(\widehat{\gamma}_1 + \widehat{\gamma}_2, \widehat{\gamma}_3) &= \Psi(\widehat{\gamma}_1, \widehat{\gamma}_3)\Psi(\widehat{\gamma}_2, \widehat{\gamma}_3)\end{aligned}$$

we deform the multiplicative unitary W^Ψ and introduce a certain deformation of $\widehat{\mathbb{G}}$ that is a good candidate for a dual of \mathbb{G}^Ψ .

To a bicharacter Ψ we assign a group homomorphism $\tilde{\Psi} : \widehat{\Gamma} \rightarrow \Gamma$ uniquely characterized by

$$\langle \widehat{\gamma}', \tilde{\Psi}(\widehat{\gamma}) \rangle = \Psi(\widehat{\gamma}', \widehat{\gamma}).$$

The unitary family $U_{\widehat{\gamma}}$ entering (3.1) is in this case given by $U_{\widehat{\gamma}} = \lambda_{\tilde{\Psi}(\widehat{\gamma})}$ and we have

$$\widehat{\rho}_\gamma^\Psi(b) = \lambda_{\tilde{\Psi}(\widehat{\gamma})}^* \widehat{\rho}_{\widehat{\gamma}}(b) \lambda_{\tilde{\Psi}(\widehat{\gamma})}.$$

Applying Rieffel functor \mathcal{RD}^Ψ to (A, ρ) we get (A^Ψ, ρ) and applying it to $\Delta \in \text{Mor}_\Gamma(A, A \otimes A)$ we get $\Delta^\Psi \in \text{Mor}_\Gamma(A^\Psi, (A \otimes A)^\Psi)$. Using (3.3) we obtain

$$\Delta^\Psi \in \text{Mor}_\Gamma(A^\Psi, A^\Psi \otimes_\Psi A^\Psi)$$

In the formulation of the next proposition we use notation introduced in Remark 3.5.

Proposition 4.6. *Let $\Delta^\Psi \in \text{Mor}(A^\Psi, A^\Psi \otimes_\Psi A^\Psi)$ be the morphisms introduced above. Then*

$$(\Delta^\Psi \otimes_\Psi \text{id}) \circ \Delta^\Psi = (\text{id} \otimes_\Psi \Delta^\Psi) \circ \Delta^\Psi \quad (4.3)$$

Proof. Identity (4.3) is obtained by application of the Rieffel functor \mathcal{RD}^Ψ to $(\Delta \otimes \text{id}), (\text{id} \otimes \Delta) \in \text{Mor}_\Gamma(A, A^{\otimes 3})$, $\Delta \in \text{Mor}_\Gamma(A, A \otimes A)$:

$$\begin{aligned}(\Delta^\Psi \otimes_\Psi \text{id}) \circ \Delta^\Psi &= \mathcal{RD}^\Psi((\Delta \otimes \text{id}) \circ \Delta) \\ &= \mathcal{RD}^\Psi((\text{id} \otimes \Delta) \circ \Delta) = (\text{id} \otimes_\Psi \Delta^\Psi) \circ \Delta^\Psi\end{aligned}$$

□

Proposition 4.7. *Adopting the above notation we have*

$$[(A^\Psi \otimes_\Psi \mathbb{1}) \cdot \Delta^\Psi(A^\Psi)] = [(\mathbb{1} \otimes_\Psi A^\Psi) \cdot \Delta^\Psi(A^\Psi)] = A^\Psi \otimes_\Psi A^\Psi$$

Proof. The proof is a straightforward application of Lemma 3.4 of [3] to $A \otimes 1, \Delta(A) \subset A \otimes A$ and $1 \otimes A, \Delta(A) \subset A \otimes A$ accordingly. □

Let ρ^A be an action of Γ on $\mathbb{G} = (A, \Delta)$ by automorphisms. Using Remark 4.4 we get the action $\rho^{\widehat{A}}$ of Γ on $\widehat{\mathbb{G}}$ by automorphisms. Let $\Phi \in \mathcal{COC}_2(\widehat{\Gamma})$ be the flip of Ψ

$$\Phi(\widehat{\gamma}_1, \widehat{\gamma}_2) = \Psi(\widehat{\gamma}_2, \widehat{\gamma}_1).$$

Applying Rieffel deformation \mathcal{RD}^Φ to $(\hat{A}, \rho^{\hat{A}})$ we get $\hat{\mathbb{G}}^\Phi = (\hat{A}^\Phi, \hat{\Delta}^\Phi)$. It turns out that certain aspects of $\hat{\mathbb{G}} - \mathbb{G}$ duality have its counterpart on the $\hat{\mathbb{G}}^\Phi - \mathbb{G}^\Psi$ level. In order to explain this let us recall that multiplicative unitary $W \in B(H \otimes H)$ may be viewed as the bicharacter $W \in M(\hat{A} \otimes A)$:

$$\begin{aligned} (\text{id} \otimes \Delta)W &= W_{12}W_{13} \\ (\hat{\Delta} \otimes \text{id})W &= W_{23}W_{13} \end{aligned}$$

Moreover slicing W with the normal functionals we recover A and \hat{A} :

$$\begin{aligned} A &= \{(\omega \otimes \text{id})W : \omega \in B(H)_*\}^{\text{cls}} \\ \hat{A} &= \{(\text{id} \otimes \omega)W : \omega \in B(H)_*\}^{\text{cls}} \end{aligned} \quad (4.4)$$

In what follows we shall construct certain unitary element W^Ψ satisfying the bicharacter identity and a counterpart of (4.4). Noting that

$$\begin{aligned} \Psi &\in M(C^*(\Gamma) \otimes C^*(\Gamma)) \subset M(\Gamma \ltimes \hat{A} \otimes \Gamma \ltimes A) \\ W &\in M(\hat{A} \otimes A) \subset M(\Gamma \ltimes \hat{A} \otimes \Gamma \ltimes A) \end{aligned}$$

we define

$$W^\Psi = \Psi W \Psi^* \in M(\Gamma \ltimes \hat{A} \otimes \Gamma \ltimes A) \quad (4.5)$$

Representing $\Gamma \ltimes A$ and $\Gamma \ltimes \hat{A}$ on $L = L^2(\Gamma) \otimes L^2(\mathbb{G})$ as described in Remark 2.3 we may view W^Ψ as an operator on $L \otimes L$. The algebra of compact operators on L is denoted by $\mathcal{K}(L)$.

Theorem 4.8. *Adopting the above notation we have*

$$A^\Psi = \{(\omega \otimes \text{id})(W^\Psi) : \omega \in B(L)_*\}^{\text{cls}}. \quad (4.6)$$

Proof. We begin by checking the Landstad conditions (2.1) for $(\omega \otimes \text{id})(W^\Psi)$. In order to check $(\text{id} \otimes \hat{\rho}^\Psi)$ -invariance of W^Ψ note that

$$\begin{aligned} (\text{id} \otimes \rho_\gamma^\Psi)(\Psi) &= \Psi(\lambda_{\hat{\Psi}(\gamma)} \otimes \mathbb{1}) \\ (\text{id} \otimes \rho_\gamma^\Psi)(W) &= (\mathbb{1} \otimes \lambda_{\hat{\Psi}(\gamma)})^* W (\mathbb{1} \otimes \lambda_{\hat{\Psi}(\gamma)}) \\ (\mathbb{1} \otimes \lambda_{\hat{\Psi}(\gamma)})^* W (\mathbb{1} \otimes \lambda_{\hat{\Psi}(\gamma)}) &= (\lambda_{\hat{\Psi}(\gamma)} \otimes \mathbb{1})^* W (\lambda_{\hat{\Psi}(\gamma)} \otimes \mathbb{1}) \end{aligned}$$

where the last equality follows from $\rho^A - \rho^{\hat{A}}$ relation (4.1)

$$(\text{id} \otimes \rho_\gamma^A)W = (\rho_\gamma^{\hat{A}} \otimes \text{id})W.$$

We compute

$$\begin{aligned} (\text{id} \otimes \hat{\rho}_\gamma^\Psi)(\Psi W \Psi^*) &= \Psi(\lambda_{\hat{\Psi}(\gamma)} \otimes \text{id})(\text{id} \otimes \lambda_{\hat{\Psi}(\gamma)})^* W (\lambda_{\hat{\Psi}(\gamma)} \otimes \text{id})^* (\text{id} \otimes \lambda_{\hat{\Psi}(\gamma)}) \Psi^* \\ &= \Psi(\rho_{\hat{\Psi}(\gamma)}^{\hat{A}} \otimes \rho_{-\hat{\Psi}(\gamma)}^A)(W) \Psi^* = W^\Psi \end{aligned}$$

Checking that the map $\Gamma \ni \gamma \mapsto \lambda_\gamma(\omega \otimes \text{id})(W^\Psi)\lambda_\gamma^*$ is norm continuous (which is the second Landstad condition) boils down to the simple computation

$$\lambda_\gamma(\omega \otimes \text{id})(W^\Psi)\lambda_\gamma^* = (\lambda_\gamma^* \cdot \omega \cdot \lambda_\gamma \otimes \text{id})(W^\Psi)$$

and the fact that the map $\Gamma \ni \gamma \mapsto \lambda_\gamma^* \cdot \omega \cdot \lambda_\gamma \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*$ is norm continuous.

In order to prove the third Landstad condition,

$$x(\omega \otimes \text{id})(W^\Psi)y \in \Gamma \ltimes A \text{ for } x, y \in C^*(\Gamma) \text{ and } \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*$$

we are reasoning as in the proof of Theorem 4.4 of [3]. Actually the argument shows that

$$A^\Psi = \{(\omega \otimes \text{id})(W^\Psi) : \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*\}^{\text{cls}}.$$

We first define

$$\mathcal{V} = \{x_1 \cdot (\omega \otimes \text{id})(W^\Psi) \cdot x_2 : \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*, x_1, x_2 \in C^*(\Gamma)\}^{\text{cls}}$$

and we note that

$$\begin{aligned}
\mathcal{V} &= \{x_1 \cdot (y_2 \cdot \omega \cdot y_1 \otimes \text{id})(W^\Psi) \cdot x_2 : \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*, x_1, x_2, y_1, y_2 \in C^*(\Gamma)\}^{\text{cls}} \\
&= \{(\omega \otimes \text{id})((y_1 \otimes x_1)\Psi W^\Psi(y_2 \otimes x_2)) : \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*, x_1, x_2, y_1, y_2 \in C^*(\Gamma)\}^{\text{cls}} \\
&= \{(\omega \otimes \text{id})((y_1 \otimes x_1)W(y_2 \otimes x_2)) : \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*, x_1, x_2, y_1, y_2 \in C^*(\Gamma)\}^{\text{cls}} \\
&= \overline{C^*(\Gamma)A C^*(\Gamma)}^{\|\cdot\|} = \Gamma \ltimes A
\end{aligned}$$

Using Lemma 2.6 of [3] we conclude (4.6). \square

The embeddings of A^Ψ into the first and the second leg of $A^\Psi \otimes_\Psi A^\Psi$ are denoted $\iota_1^{A^\Psi}, \iota_2^{A^\Psi}$ respectively:

$$\begin{aligned}
\iota_1^{A^\Psi}(a) &= a \otimes_\Psi \mathbf{1} \\
\iota_2^{A^\Psi}(a) &= \mathbf{1} \otimes_\Psi a
\end{aligned}$$

for any $a \in A^\Psi$. Note that equality (4.6) does not automatically imply that $W^\Psi \in M(\mathcal{K}(L) \otimes A^\Psi)$.

Theorem 4.9. *Let W^Ψ be the operator defined in (4.5). Then*

$$(\text{id} \otimes (\Gamma \ltimes \Delta))(W^\Psi) = (\text{id} \otimes \Gamma \ltimes \iota_1^A)(W^\Psi)(\text{id} \otimes \Gamma \ltimes \iota_2^A)(W^\Psi) \quad (4.7)$$

In particular if $W^\Psi \in M(\mathcal{K}(L) \otimes A^\Psi)$ then

$$(\text{id} \otimes \Delta^\Psi)(W^\Psi) = (\text{id} \otimes \iota_1^{A^\Psi})(W^\Psi)(\text{id} \otimes \iota_2^{A^\Psi})(W^\Psi) \quad (4.8)$$

Proof. In order to prove (4.7) we compute

$$\begin{aligned}
(\text{id} \otimes \Gamma \ltimes \Delta)(\Psi W^\Psi)^* &= (\text{id} \otimes \Gamma \ltimes \Delta)(\Psi)((\text{id} \otimes \Delta)W)(\text{id} \otimes \Gamma \ltimes \Delta)(\Psi)^* \\
&= (\text{id} \otimes \Gamma \ltimes \Delta)(\Psi)W_{12}W_{13}(\text{id} \otimes \Gamma \ltimes \Delta)(\Psi)^* \\
&= (\text{id} \otimes \Gamma \ltimes \iota_1^A)(\Psi)W_{12}W_{13}(\text{id} \otimes \Gamma \ltimes \iota_2^A)(\Psi)
\end{aligned}$$

In the last step we used the equality of three respective restrictions

$$\Gamma \ltimes \Delta|_{C^*(\Gamma)} = \Gamma \ltimes \iota_1^A|_{C^*(\Gamma)} = \Gamma \ltimes \iota_2^A|_{C^*(\Gamma)}$$

all of them coinciding with the standard embedding $C^*(\Gamma) \subset M(\Gamma \ltimes (A \otimes A))$. Noting that

$$\begin{aligned}
(\text{id} \otimes \Gamma \ltimes \iota_1^A)(W) &= W_{12} \\
(\text{id} \otimes \Gamma \ltimes \iota_2^A)(W) &= W_{13}
\end{aligned}$$

we get

$$\begin{aligned}
(\text{id} \otimes \Delta^\Psi)(W^\Psi) &= (\text{id} \otimes \Gamma \ltimes \Delta)(\Psi W^\Psi)^* \\
&= (\text{id} \otimes \Gamma \ltimes \Delta)(\Psi)W_{12}W_{13}(\text{id} \otimes \Gamma \ltimes \Delta)(\Psi)^* \\
&= (\text{id} \otimes \Gamma \ltimes \iota_1^A)(\Psi W^\Psi)^*(\text{id} \otimes \Gamma \ltimes \iota_2^A)(\Psi W^\Psi)^*
\end{aligned}$$

Since $\Gamma \ltimes \Delta|_{A^\Psi} = \Delta^\Psi$, $\Gamma \ltimes \iota_1^A|_{A^\Psi} = \iota_1^{A^\Psi}$ and $\Gamma \ltimes \iota_2^A|_{A^\Psi} = \iota_2^{A^\Psi}$ then with the assumption that $W^\Psi \in M(\mathcal{K}(L) \otimes A^\Psi)$ we get (4.8). \square

Since the multiplicative unitary \widehat{W} for $\widehat{\mathbb{G}}$ is given by $\Sigma W^* \Sigma$ we immediately get the analogous theorem for $(\widehat{A}^\Phi, \widehat{\Delta}^\Phi)$. In particular

$$\widehat{A}^\Phi = \{(\text{id} \otimes \omega)W^\Psi : \omega \in B(L^2(\Gamma) \otimes L^2(\mathbb{G}))_*\}^{\text{cls}}.$$

Let us emphasize that we were not able to conclude that $W^\Psi \in M(\mathcal{K}(L) \otimes A^\Psi)$. Assuming this and $W^\Psi \in M(\widehat{A}^\Phi \otimes \mathcal{K}(L))$ we were not able to conclude that $W^\Psi \in M(\widehat{A}^\Phi \otimes A^\Psi)$. In the quantum group theory the respective results are obtained by the pentagonal equation which is not available for W^Ψ .

5. ON A CERTAIN EXAMPLE OF A QUANTUM MINKOWSKI SPACE

In this section we employ Rieffel deformation to get an example of a C^* -quantum Minkowski space. In order to do it we elaborate the example described in [4] where Rieffel deformation functor \mathcal{RD}^Ψ was applied to the C^* -algebra $C_0(\mathcal{M})$ of the Minkowski space \mathcal{M} . Since in the context of this example the Minkowski space was equipped with the action of $SL(2, \mathbb{C})$ (the twofold cover of the connected component of the Lorentz group) we identify \mathcal{M} with the space of selfadjoint matrices \mathcal{H}

$$\mathcal{H} = \left\{ \begin{bmatrix} x & w \\ \bar{w} & y \end{bmatrix} : x, y \in \mathbb{R}, w \in \mathbb{C} \right\} \quad (5.1)$$

by the map

$$\mathcal{M} \ni [x_\mu] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \sigma([x_\mu]) = \begin{bmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{bmatrix} \in \mathcal{H}.$$

With this identification the action of $SL(2, \mathbb{C})$ on \mathcal{M} is given by matrix conjugation

$$\sigma(\alpha_g[x_\mu]) = g\sigma([x_\mu])g^*$$

Equipping \mathcal{M} with the addition

$$[x_\mu] + [y_\mu] = [x_\mu + y_\mu]$$

we view α_g as the (Lorentz metric preserving) automorphisms of \mathcal{M} .

On the C^* -level we get $\mathbb{M} = (C_0(\mathcal{M}), \Delta)$ where for $f \in C_0(\mathcal{M})$

$$\Delta(f)([x_\mu], [y_\mu]) = f([x_\mu + y_\mu]).$$

Using the restriction of α to the subgroup of diagonal matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{C} \right\} \subset SL(2, \mathbb{C})$$

we get the data needed to perform Rieffel deformation of \mathbb{M} . For convenience we pull back the action to \mathbb{C} by the group homomorphism:

$$\mathbb{C} \ni z \mapsto \begin{bmatrix} e^z & 0 \\ 0 & e^{-z} \end{bmatrix} \in \Gamma$$

and we define

$$\rho : \mathbb{C} \rightarrow \text{Aut}(\mathbb{M}) : (\rho_z f)([x_\mu]) = f(\alpha_{\sigma(z)}[x_\mu])$$

We identify \mathbb{C} and $\widehat{\mathbb{C}}$ by the duality

$$\langle z_1, z_2 \rangle = \exp(i\Im(z_1 z_2))$$

and define a bicharacter on $\widehat{\mathbb{C}}$:

$$\Psi(z_1, z_2) = \exp(is\Im(z_1 \overline{z_2}))$$

$s \in \mathbb{R}$ being the deformation parameter.

Let $\mathbb{M}^\Psi = (C_0(\mathcal{M})^\Psi, \Delta^\Psi)$ be the Rieffel deformation of \mathbb{M} . In order to construct the \mathbb{M}^Ψ -counterparts of x, y, w coordinates let us consider $V \in M(C^*(\mathbb{C}))$ such that $V(z) = \exp(i\frac{s}{2}\Im(z^2))$ (note the $C^*(\mathbb{C}) \cong C_0(\mathbb{C})$ identification). We define

$$\widehat{x} = e^{-2s} V x V^*, \widehat{y} = e^{-2s} V y V^*, \widehat{w} = e^{2s} V^* w V \quad (5.2)$$

which are affiliated elements of $\mathbb{C} \rtimes C_0(\mathcal{M})$.

For the concept of an element T affiliated with a C^* -algebra A , $T \eta A$ we refer to [15] where an idea of C^* -algebras generated by affiliated elements is also developed. In Theorem 5.3 of [4] we

proved that $\widehat{x}, \widehat{y}, \widehat{w} \in C_0(\mathcal{M})^\Psi$ and C^* -algebra $C_0(\mathcal{M})^\Psi$ is generated by $\widehat{x}, \widehat{y}, \widehat{w}$. Remarkably $\widehat{x}, \widehat{y}, \widehat{w}$ satisfy the following commutation relations

$$\begin{aligned}\widehat{x}\widehat{w} &= t^{-1}\widehat{w}\widehat{x} \\ \widehat{x}\widehat{w}^* &= t\widehat{w}^*\widehat{x} \\ \widehat{y}\widehat{w} &= t\widehat{w}\widehat{y} \\ \widehat{y}\widehat{w}^* &= t\widehat{w}^*\widehat{y}\end{aligned}\tag{5.3}$$

where $t = e^{-8s}$. For the precise meaning of (5.3) we advice to consult Theorem 3.7 of [4] where the meaning of the relation

$$\begin{aligned}RS &= p^2SR \\ RS^* &= q^2S^*R\end{aligned}$$

for normal operators R, S is explained. The Hilbert space version of (p, q) -commutation relation was introduced in [14].

In the formulation of the next theorem where we analyze the action of Δ^Ψ on generators we use the extension of notation (3.4) to affiliated elements writing

$$T \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi T \in A^\Psi \otimes_\Psi A^\Psi$$

for $T \in A^\Psi$.

Theorem 5.1. *Let $\widehat{x}, \widehat{y}, \widehat{w} \in C_0(\mathcal{M})^\Psi$ be the affiliated elements defined above. Then*

- *$(\widehat{x} \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi \widehat{x}), (\widehat{y} \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi \widehat{y}), (\widehat{w} \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi \widehat{w})$ and $(\widehat{x} \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi \widehat{y})$ are strongly commuting pairs of normal elements affiliated with $C_0(\mathcal{M})^\Psi \otimes_\Psi C_0(\mathcal{M})^\Psi$ and we have*

$$\begin{aligned}\Delta_{\mathcal{M}}^\Psi(\widehat{x}) &= \widehat{x} \otimes_\Psi \mathbf{1} + \mathbf{1} \otimes_\Psi \widehat{x} \\ \Delta_{\mathcal{M}}^\Psi(\widehat{y}) &= \widehat{y} \otimes_\Psi \mathbf{1} + \mathbf{1} \otimes_\Psi \widehat{y} \\ \Delta_{\mathcal{M}}^\Psi(\widehat{w}) &= \widehat{w} \otimes_\Psi \mathbf{1} + \mathbf{1} \otimes_\Psi \widehat{w}\end{aligned}\tag{5.4}$$

- *$(\widehat{x} \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi \widehat{w})$ and $(\widehat{y} \otimes_\Psi \mathbf{1}, \mathbf{1} \otimes_\Psi \widehat{w})$ are respectively (t^{-1}, t) and (t, t^{-1}) -commuting pair of normal elements affiliated with $C_0(\mathcal{M})^\Psi \otimes_\Psi C_0(\mathcal{M})^\Psi$.*

Proof. Note that

$$\begin{aligned}\widehat{x} \otimes_\Psi \mathbf{1} &= e^{-2s}U(x \otimes \mathbf{1})U^* \\ \mathbf{1} \otimes_\Psi \widehat{x} &= e^{-2s}U(\mathbf{1} \otimes x)U^* \\ \widehat{y} \otimes_\Psi \mathbf{1} &= e^{-2s}U(y \otimes \mathbf{1})U^* \\ \mathbf{1} \otimes_\Psi \widehat{y} &= e^{-2s}U(\mathbf{1} \otimes y)U^* \\ \widehat{w} \otimes_\Psi \mathbf{1} &= e^{2s}U(w \otimes \mathbf{1})U^* \\ \mathbf{1} \otimes_\Psi \widehat{w} &= e^{2s}U(\mathbf{1} \otimes w)U^*\end{aligned}$$

where U is the image of $V = \exp(\imath \frac{s}{2} \Im(z^2)) \in M(C^*(\mathbb{C}))$ under the embedding

$$\iota^{C^*(\Gamma)} \in \text{Mor}(C^*(\mathbb{C}), \mathbb{C} \ltimes (C_0(\mathcal{M}) \otimes C_0(\mathcal{M}))).$$

Noting that

- the commutation relation between U and $x \otimes \mathbf{1}, y \otimes \mathbf{1}$ and $w \otimes \mathbf{1}$ are the same as in the case of V and x, y, w
- the commutation relation between U and $\mathbf{1} \otimes x, \mathbf{1} \otimes y$ and $\mathbf{1} \otimes w$ are the same as in the case of V and x, y, w

the techniques of the proof of Theorem 5.3 of [4] may be applied giving the commutation relation of our theorem.

Let us compute $\Delta^\Psi(\widehat{x})$:

$$\begin{aligned}\Delta^\Psi(\widehat{x}) &= (\mathbb{C} \ltimes \Delta)(e^{-2s}VxV^*) \\ &= e^{-2s}U(x \otimes 1 + 1 \otimes x)U^* \\ &= e^{-2s}U(x \otimes 1)U^* + e^{-2s}U(1 \otimes x)U^* \\ &= \widehat{x} \otimes_\Psi \mathbf{1} + \mathbf{1} \otimes_\Psi \widehat{x}\end{aligned}$$

Similarly we get the remaining identities of (5.4). □

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